

Kurt Gödel to John von Neumann

A polished English translation

Princeton, 20 March 1956

Dear Mr. von Neumann,

I was deeply saddened to learn of your illness. The news came as a complete surprise to me. Morgenstern had told me last summer about an attack of weakness you had suffered, but at the time he thought it was of no great significance. I understand that during the past few months you have undergone a radical treatment, and I am glad that it has had the desired success and that you are now feeling better. I hope and wish that your condition will soon improve still further, and that the latest advances in medicine may, if possible, bring about a complete recovery.

Since, as I hear, you are now feeling stronger, I would like to take the liberty of writing to you about a mathematical problem on which I would be very interested to know your opinion. It is evidently easy to construct a Turing machine which, given any formula F of first-order predicate calculus and any natural number n , determines whether F has a proof of length n , where length means the number of symbols. Let $\Psi(F, n)$ be the number of steps the machine requires, and let

$$\varphi(n) = \max_F \Psi(F, n).$$

The question is how rapidly $\varphi(n)$ grows for an optimal machine. One can show that

$$\varphi(n) \geq kn.$$

If there really were a machine for which $\varphi(n) \sim kn$, or even merely $\varphi(n) \sim kn^2$, the consequences would be of the greatest significance. It would evidently mean that, despite the undecidability of the decision problem, the intellectual work of the mathematician on yes-or-no questions could be entirely replaced by machines, apart from formulating the axioms. One would simply have to choose n so large that, if the machine produced no result, there would be no point in thinking further about the problem.

It seems to me, however, entirely possible that $\varphi(n)$ grows this slowly. For, first, $\varphi(n) \geq kn$ appears to be the only estimate obtainable by generalizing the proof of the undecidability of the decision problem. Second, $\varphi(n) \sim kn$, or $\varphi(n) \sim kn^2$, means only that the number of steps, compared with mere trial, can be reduced from N to $\log N$, or to $(\log N)^2$. Reductions of this magnitude do occur in other finite problems, for example in calculating a quadratic residue symbol by repeated application of the law of quadratic reciprocity. It would be interesting to know how matters stand, for instance, in determining whether a number is prime, and more generally how far the number of steps in finite combinatorial problems can be reduced relative to brute-force trial.

I do not know whether you have heard that "Post's problem"—whether there are intermediate degrees of unsolvability among problems of the form $(\exists y) \varphi(y, x)$, with recursive φ —has been solved affirmatively by a very young man named Richard Friedberg. The solution is very elegant. Unfortunately, Friedberg intends to study medicine rather than mathematics, apparently under the influence of his father.

Incidentally, what do you think of the attempts to base analysis on ramified type theory, which have recently come back into fashion? You probably know that Paul Lorenzen has carried this approach as far as the theory of Lebesgue measure. But I believe that important parts of analysis involve impredicative methods of reasoning that cannot be eliminated.

I would be very glad to hear from you personally. Please let me know if there is anything I can do for you.

With my best regards and wishes, also to your wife,

Yours sincerely,

Kurt Gödel

P.S. My warmest congratulations on the honor conferred upon you by the American government.

John von Neumann, who received the Presidential Medal of Freedom in 1956, had cancer at the time of the letter and passed away in 1957.