

Universal Sequential Search Problems

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Abstract

The article examines several well-known problems of the “sequential search type” and proves that these problems can only be solved in the time it takes to solve any problem of the specified type in general.

1 Introduction

[Introduction remains the same as in the previous version]

2 Definitions and Problems

[Definitions and Problems section remains the same as in the previous version]

3 Main Results

Let $f(n)$ be a monotonic function.

Theorem 1. *If there exists any problem of sequential search (quasi-sequential search) type that cannot be solved in time less than $f(n)$ for argument length comparable to n , then problems 1-6 also have this property.*

Proof. Let P be a sequential search problem that cannot be solved in time less than $f(n)$ for argument length comparable to n . We will show that each of the problems 1-6 is at least as hard as P .

By Lemma 1 (proven below), problems 1-6 are universal sequential search problems. This means that P can be reduced to each of these problems. Let's consider the reduction of P to problem i (where i is any number from 1 to 6).

There exist three algorithms $r(x)$, $p(y)$, and $s(y)$, working in time comparable to the length of the argument, such that:

1) $P(x, p(y)) \Leftrightarrow \text{Problem}_i(r(x), y)$ 2) $P(x, y) \Leftrightarrow \text{Problem}_i(r(x), s(y))$

Suppose, for contradiction, that problem i can be solved in time less than $f(n)$. Then we could solve P as follows:

1) Given input x for P , compute $r(x)$. 2) Solve $\text{Problem}_i(r(x), y)$ in time less than $f(n)$. 3) If a solution y is found, compute $p(y)$ to get a solution for P .

The total time for this procedure would be less than $f(n)$ (plus some time comparable to the input length for computing $r(x)$ and $p(y)$). This contradicts our assumption that P cannot be solved in time less than $f(n)$.

Therefore, problem i cannot be solved in time less than $f(n)$. Since i was arbitrary, this applies to all problems 1-6. \square

The idea of the proof is that problems 1-6 are “universal sequential search problems”.

Definition 1. Let $A(x, y)$ and $B(x, y)$ define sequential search problems A and B respectively. We say that problem A reduces to B if there are three algorithms $r(x)$, $p(y)$, and $s(y)$, working in time comparable to the length of the argument, such that $A(x, p(y)) \Leftrightarrow B(r(x), y)$ and $A(x, y) \Leftrightarrow B(r(x), s(y))$ (i.e., from an A -problem x , it's easy to construct an equivalent B -problem $r(x)$). A problem to which any sequential search problem reduces is called “universal”.

Thus, the essence of the proof of Theorem 1 consists in the following lemma.

Lemma 1. Problems 1-6 are universal sequential search problems.

Proof Sketch. We need to show that any sequential search problem can be reduced to each of the problems 1-6. We'll outline the reduction for Problem 2 (finding a DNF for a partial Boolean function).

Let $A(x, y)$ be any sequential search problem. We can encode the computation of $A(x, y)$ as a Boolean circuit. This circuit can be represented as a partial Boolean function f_x where:

- The input represents y - The output is 1 if and only if $A(x, y)$ is true - The function is undefined for inputs that don't correspond to valid encodings of y

Now, finding a y such that $A(x, y)$ is true is equivalent to finding a satisfying assignment for f_x , which in turn is equivalent to finding a DNF representation of f_x .

The reduction algorithms would work as follows: - $r(x)$ constructs the partial Boolean function f_x - $p(y)$ extracts y from the satisfying assignment - $s(y)$ encodes y as an input to the Boolean function

Similar reductions can be constructed for the other problems, showing that each of them is universal. \square

The described method apparently allows easy obtaining of results like Theorem 1 and Lemma 1 for most interesting sequential search problems. However, the problem remains to prove the condition present in this theorem. Numerous attempts have long been made in this direction, and a number of interesting results have been obtained (see, for example, [3, 4]). However, the universality of various sequential search problems can be established without solving this problem. In the system of Kolmogorov-Uspensky algorithms, the following can also be proved:

Theorem 2. For an arbitrary sequential search problem $A(x, y)$, there exists an algorithm that solves it in time optimal to within multiplication by a constant and addition of a value comparable to the length of x .

Proof Sketch. The idea is to construct a "universal" algorithm that simulates all possible algorithms in parallel, allocating more and more time to each algorithm as the computation progresses.

Let $\{M_i\}$ be an enumeration of all algorithms (e.g., all Turing machines). We construct an algorithm U that works as follows:

For $t = 1, 2, 3, \dots$: For $i = 1, 2, \dots, t$: Run M_i on input x for $2^{i-t}t$ steps. If M_i outputs a y such that $A(x, y)$ is true, return y .

If there exists an algorithm that solves $A(x, y)$ in time $T(n)$, then U will find a solution in time $O(T(n))$. The multiplicative constant comes from the overhead of simulating multiple machines, and the additive term comparable to the length of x comes from the initial steps where t is small.

This algorithm is optimal up to a constant factor because if there were a significantly faster algorithm, it would contradict the assumption that $T(n)$ was the time of the fastest algorithm. \square

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