The Infinite Version of an Open Communication Complexity Problem is Independent of the Axioms of Set Theory

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Abstract

In 1986, Babai, Frankl and Simon [BFS86] defined the polynomial hierarchy in communication complexity and asked whether $\Sigma_2^{cc} = \Pi_2^{cc}$. In order to tackle this problem, researchers have looked at an infinite version. We recently became aware of a paper from 1979 where Miller [Mil79] shows that this infinite version is independent of the axioms of set theory. In this note we will describe Miller's result and give a simplified proof of one direction by showing that the continuum hypothesis implies that $\Sigma_2^r = \Pi_2^r = \mathcal{P}(\mathbb{R} \times \mathbb{R})$.

One approach to solving problems in complexity theory is to look at infinite versions of problems where the solutions may be easier. One can then try to apply these proof techniques to the finite complexity theory question. In one of the best examples of this technique, Sipser (see [Sip83]) showed that an infinite version of parity does not have bounded depth countable-size circuits. Furst, Saxe and Sipser [FSS84] used the techniques from Sipser's paper to show that parity does not have constant-depth polynomial-size circuits.

In 1986, Babai, Frankl and Simon [BFS86] defined a hierarchy of communication complexity classes and asked whether $\Sigma_2^{cc} = \Pi_2^{cc}$ in this hierarchy. In this paper we will look at the equivalent combinatorial definition of this hierarchy. We refer the reader to [BFS86] for a background in communication complexity and the communication complexity definition of this hierarchy.

Let $N = \{1, \ldots 2^n\}$. A *rectangle* is defined to be $A \times B$ where A and B are *arbitrary* subsets of $N \times N$. Let Π_0^{cc} be the set of rectangles. For every $i \geq 0$, let Σ_{i+1}^{cc} be any polynomial union of Π_i^{cc} sets and let Π_{i+1}^{cc} be the set of complements of Σ_{i+1}^{cc} sets. Note that the diagonal set $\{(x,x) \mid x \in N\}$ is in $\Pi_1^{cc} - \Sigma_1^{cc}$.

In order to tackle the $\Sigma_2^{cc} = \Pi_2^{cc}$ problem, researchers have looked at an infinite version of the communication complexity hierarchy. We define the Σ_i^r, Π_i^r levels of the rectangle hierarchy the same way as the communication complexity hierarchy except we use $\mathbb{R}$ instead of N and countable union instead of polynomial union. Note again that the diagonal set $\{(x,x) \mid x \in \mathbb{R}\}$ is in $\Pi_1^r - \Sigma_1^r$.

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We recently became aware of a 1979 paper by Miller [Mil79] that settles the $\Sigma_2^r = \Pi_2^r$ question in an unusual way: "$\Sigma_2^r = \Pi_2^r$" is independent of the standard axioms of set theory. As far as we know, this is the first known independence result for an infinite version of a complexity question.

Note that the independence of the infinite case does not imply anything about the provability of the communication complexity question. What it does tell us is that any proof that $\Sigma_2^{cc} = \Pi_2^{cc}$ or $\Sigma_2^{cc} \neq \Pi_2^{cc}$ will not carry over to the infinite case.

As we shall see, if one assumes the axiom of choice and the continuum hypothesis (there is no set smaller than the reals but bigger than the natural numbers) then we have that $\Sigma_2^r = \Pi_2^r$ equals every subset of $\mathbb{R} \times \mathbb{R}$. On the other hand, Miller shows that it is consistent with the axioms of set theory that the rectangle hierarchy collapse at precisely the α level for α any successor ordinal greater than one and less than ω_1.

The proof that the continuum hypothesis and the axiom of choice imply that $\Sigma_2^r = \Pi_2^r$ equals every subset of $\mathbb{R} \times \mathbb{R}$ can be understood without a deep knowledge of set theory. In this note we will present a simplified proof of this fact.

First some definitions and notations:

Let ZFC be the standard axioms of set theory including the axiom of choice (AC).

Let $\omega = \omega_0$ be the set of natural numbers. Let ω_1 be the smallest ordinal with uncountable cardinality. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Q}$ denote the set of rational numbers.

Let $|X|$ denote the cardinality of X and $\mathcal{P}(X)$ and 2^X denote the power set of X, i.e., the set of all subsets of X.

The continuum hypothesis (CH) states that $|\omega_1| = |2^\omega| (= |\mathbb{R}|)$.

Theorem 1 *ZFC implies* $\mathcal{P}(\omega_1 \times \omega_1) \subseteq \Sigma_2^r$

Proof: Let T be any subset of $\omega_1 \times \omega_1$. We will show that T is a Σ_2^r set, i.e., a countable union of Π_1^r sets.

Note that for every $\alpha < \omega_1$ there are at most a countable number of $\beta \leq \alpha$. For every $\alpha < \omega_1$ define A_α and B_α as

$$A_\alpha = \{(\beta, \alpha) \mid (\beta, \alpha) \in T \ \& \ \beta \leq \alpha\}$$

$$B_\alpha = \{(\alpha, \beta) \mid (\alpha, \beta) \in T \ \& \ \beta < \alpha\}$$

Note that for every $\alpha < \omega_1$, A_α and B_α are countable and

$$T = \bigcup_{\alpha < \omega_1} A_\alpha \cup B_\alpha$$

Since A_α and B_α are countable, fix some enumeration of their elements using the axiom of choice.

Define the partial function $f_A : \omega_1 \times \omega \to \omega_1$ as follows:

$f_A(\alpha, i) = \beta$ where (β, α) is the ith element in the enumeration of A_α if an ith element exists.

Define S_A as:

$$S_A(i) = \bigcup_{\alpha < \omega_1} \{(f(\alpha, i), \alpha) \mid f(\alpha, i) \text{ is defined}\}$$

We can define f_B and S_B similarly.

Note that

$$T = \bigcup_{i \in \omega} (S_A(i) \cup S_B(i))$$

We now only need to show that for every $i \in \omega$, $S_A(i)$ and $S_B(i)$ are both Π^r_1 sets.

Fix $i \in \omega$. Since $|\omega_1| \leq |2^\omega| = |\mathbb{R}|$ there exists a one-to-one function g that maps ω_1 into $\mathbb{R}$.

Define rectangles Θ and Δ_q and Γ_q for $q \in \mathbb{Q}$ as follows:

$$\begin{aligned}
\Theta &= \{(\beta, \alpha) \mid f_A(\alpha, i) \text{ is not defined}\} \\
\Delta_q &= \{(\beta, \alpha) \mid f_A(\alpha, i) \text{ is defined and } g(\beta) < q \text{ and } g(f_A(\alpha, i)) > q\} \\
\Gamma_q &= \{(\beta, \alpha) \mid f_A(\alpha, i) \text{ is defined and } g(\beta) > q \text{ and } g(f_A(\alpha, i)) < q\}
\end{aligned}$$

Note that

$$\overline{S_A(i)} = \Theta \cup \bigcup_{q \in \mathbb{Q}} (\Delta_q \cup \Gamma_q)$$

because for every $(\beta, \alpha) \notin S_A(i)$, either $f_A(\alpha, i)$ is not defined or there is some $q \in Q$ such that either $g(\beta) < q < g(f_A(\alpha, i))$ or $g(\beta) > q > g(f_A(\alpha, i))$. Thus $S_A(i)$ is a Π^r_1 set. The proof that $S_B(i)$ is a Π^r_1 set is similar. □

Corollary 2 *ZFC + CH implies that* $\Sigma^r_2 = \Pi^r_2 = \mathcal{P}(\mathbb{R} \times \mathbb{R})$.

Miller notes that the converse of Corollary 2 does not hold.

Miller's proof showing that it is consistent with the axioms of set theory that $\Sigma^r_2 \neq \Pi^r_2$ uses forcing techniques requiring a substantial background in set theory. We state the appropriate theorem from Miller's paper [Mil79, Theorem 37] and refer the interested reader to that paper.

Theorem 3 *For any* α_0 *a successor ordinal such that* $2 \leq \alpha_0 < \omega_1$*, it is relatively consistent with ZFC that* $|2^\omega| = \omega_2$ *and* α_0 *is the least ordinal such that* $\Sigma^r_{\alpha_0} = \mathcal{P}(\mathbb{R} \times \mathbb{R})$.

Corollary 4 *For any* α_0 *a successor ordinal such that* $2 \leq \alpha_0 < \omega_1$*, it is relatively consistent with ZFC that* $|2^\omega| = \omega_2$ *and* α_0 *is the least ordinal such that* $\Sigma^r_{\alpha_0} = \Pi^r_{\alpha_0}$

Proof: Note that for any $\alpha < \omega_1$, if $\Sigma^r_\alpha = \Pi^r_\alpha$ then for any $\beta \geq \alpha$, $\Sigma^r_\beta = \Pi^r_\beta$.

References

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